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Quantum superalgebras at roots of unity and non-Abelian symmetries of integrable models

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Abstract

We consider integrable vertex models whose Boltzmann weights (R-matrices) are trigonometric solutions to the graded Yang–Baxter equation. As is well known, the latter can be generically constructed from quantum affine superalgebras $U_q(\hat{g})$. These algebras do not form a symmetry algebra of the model for generic values of the deformation parameter q when periodic boundary conditions are imposed. If q is evaluated at a root of unity we demonstrate that in certain commensurate sectors one can construct non-Abelian subalgebras which are translation invariant and commute with the transfer matrix and therefore with all charges of the model. In the line of argument, we introduce the restricted quantum superalgebra $U_q^{\rm res}(\hat{g})$ and investigate its root of unity limit. We prove several new formulae involving supercommutators of arbitrary powers of the Chevalley–Serre generators and derive higher order quantum Serre relations as well as an analogue of Lustzig's quantum Frobenius theorem for superalgebras.

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1. Introduction

The area of integrable models has proved to be one of the most fruitful connections between physics and mathematics over the years. Especially, the study of the Yang–Baxter equation [1, 2] has led to numerous discoveries of new algebraic structures with a range of applications reaching far beyond its origin in exactly solvable statistical mechanics models [3] and the quantum inverse scattering method [4, 5]. The widely known and best studied examples of such new structures are quantum algebras (also called quantum groups) [6–8]. The latter are obtained from an affine Lie algebra $\hat{g}$ as a particular q-deformation $U_q(\hat{g})$ and belong to the class of non-cocommutative Hopf algebras. Once a coproduct is chosen their quasi-triangular structure (the universal R-matrix) gives rise to trigonometric solutions of the Yang–Baxter equation, which have a direct physical interpretation either as the two-particle amplitude in factorizable scattering matrix theory or as the Boltzmann weights of a two-dimensional statistical lattice model. In this work, we will concentrate on the latter application.

Despite this intimate relation between solvable lattice models and quantum algebras, it is important to keep in mind that in general $U_q(\hat{g})$ does not provide a symmetry of the physical system. In fact, the integrability of the model follows from the Yang–Baxter equation alone and manifests itself in commuting transfer matrices, which form an Abelian symmetry. In contrast, the quantum algebra is non-Abelian and its generators neither commute with the transfer matrix nor with the charges of the model when periodic boundary conditions are imposed on the lattice. The latter, however, are often chosen to render the model translation invariant simplifying its physical discussion via the Bethe ansatz [9]. The connection between the Bethe ansatz and the representation theory of the quantum algebra $U_q(\hat{g})$ in the presence of periodic boundary conditions is not fully understood to date.

The situation becomes different when the deformation parameter q approaches a root of unity, $q^N \to 1$. It was Baxter [10] who first noted in the context of the XYZ model and one of its specializations, the XXZ or six-vertex model related to $U_q(\widehat{sl}_2)$, that in this case extra degeneracies in the eigenvalue spectrum of the transfer matrix appear. Subsequently, the properties of the XXZ model at roots of unity have also been investigated by several other authors [11–13].

The symmetry underlying these degeneracies remained unclear until Deguchi, Fabricius and McCoy [14, 15] showed that it can be linked to a finite-dimensional representation of the non-deformed affine algebra $\widehat{sl}_2$ in certain commensurate sectors where the spin is a multiple of the order N of the root of unity. In particular, the symmetry generators can be explicitly constructed as a subalgebra of $U_q(\widehat{sl}_2)$ and are compatible with periodic boundary conditions. As recently discussed in [16], this allows us to connect the finite-dimensional representation theory of the affine algebra with the methods of the algebraic Bethe ansatz leading to an exact formula for the dimension of the degenerate eigenspaces at arbitrary roots of unity. See also [17] for a combinatorial approach in the special case $N = 6$.

The occurrence of the loop symmetry at roots of unity is a general phenomenon as has been demonstrated in [18], is widened and generalized to arbitrary quantum affine algebras $U_q(\hat{g})$ covering to a large extent the known integrable vertex models. In this work, we extend the discussion even further by considering integrable vertex models whose R-matrix is a trigonometric solution to the graded Yang–Baxter equation [19].

While from the statistical mechanics point of view no gradation is distinguished, those algebras which carry a $\mathbb{Z}_2$-gradation and are known as superalgebras have been most thoroughly discussed in the literature, see [20, 21] and also [22] for a recent presentation. The definition of quantum affine superalgebras and their connection to integrable models has been developed in e.g. [23–32]. *A posteriori* it became clear that pre-existing vertex models such as [33, 34] belonged to this class. Additional motivation to consider models associated with quantum superalgebras has been the observation that their associated spin-chain Hamiltonians provide generalizations of the Hubbard model and describe quasi one-dimensional strongly-correlated electron systems (see e.g. [35–41] for models associated with quantum superalgebras and [42] for further references).

While we have outlined so far the physical interest in considering the root of unity case, this paper will mainly be concerned with the mathematical structure needed for the construction of the non-Abelian symmetries at roots of unity. The representation theory of quantum algebras at roots of unity entered the mathematical literature roughly twenty years after Baxter's observation of degeneracies in the XYZ, respectively XXZ, model. When $\hat{g}$ is non-graded and $q^N = 1$ there have been two different forms of the quantum algebra discussed, the non-restricted form $U_q(\hat{g})$ [43–45] and the restricted one $U_q^{\text{res}}(\hat{g})$ [46, 47]. The latter is obtained when dividing the q-deformed Chevalley–Serre generators e_i^n, f_i^n by their q-deformed powers. While for generic deformation parameter q the two realizations are equivalent, they lead to quite different structures at roots of unity $q^N = 1$, both of which have physical applications. The non-restricted form is relevant for the chiral Potts model whose original formulation [48] in the physics literature again preceded the mathematical discussion (see also [49] for the connection with cyclic representations of $U_q(\widehat{sl}_2)$). The physical importance of the restricted form $U_q^{\text{res}}(\hat{g})$ when $\hat{g}$ is non-graded has become apparent by the discussion in [14, 18] as outlined above. The symmetry generators underlying the degeneracies at roots of unity can be directly obtained as a subalgebra from $U_q^{\text{res}}(\hat{g})$.

When $\hat{g}$ is an affine superalgebra the analogous discussion of representation theory is largely missing in the literature. While the root of unity limit of the non-restricted quantum superalgebra $U_q(\hat{g})$ has been considered for some specific cases (see e.g. [27, 50–54]) the structure of the restricted algebra $U_q^{\text{res}}(\hat{g})$ has not been subject to investigations so far. In this paper, we will put forward its definition and discuss its algebraic structure. This will put us in the position to construct explicitly non-Abelian symmetry algebras of integrable models. The detailed outline of this paper is as follows.

In section 2 we shortly review the definition of superalgebras and their q-deformed counterparts in order to keep the paper self-contained. We will focus on those aspects which are relevant in hindsight of our discussion and its connection to integrable lattice models. In particular, we recall the non-commutative Hopf algebra structure which provides the natural setting for the graded Yang–Baxter equation. In section 3 we introduce analogous to the non-graded case the restricted quantum algebra $U_q^{\text{res}}(\hat{g})$ for affine superalgebras $\hat{g}$. We calculate the supercommutation relation of its elements, state their coproduct formulae and derive the analogue of Lustzig's higher order quantum Serre relations [46] for superalgebras. In section 4 we discuss in detail the root of unity limit and demonstrate that the analogue of the non-graded quantum Frobenius homomorphism $U_q^{\text{res}}(\hat{g}) \to U(\hat{g})$ at roots of unity holds at most for the even subalgebras in the super case. In section 5 we relate the previous discussion to integrable lattice models associated with quantum affine superalgebras. We in particular investigate under which conditions the generators of $U_q^{\text{res}}(\hat{g})$ in the L-fold tensor product are translation invariant when $q^N \to 1$. These findings then allow us to identify the subalgebras which form non-Abelian symmetry algebras of integrable lattice models. Section 6 contains our conclusions.

2. Quantum superalgebras

This section gives a short review of the classification of superalgebras. We will only consider contragredient or basic superalgebras, which possess a non-degenerate invariant bilinear form and are the most relevant ones for the application we have in mind. For details, we refer the reader to [20] (see also [22] for a recent presentation and further references on superalgebras). We start by recalling that superalgebras g are generalizations of Lie algebras which carry an $\mathbb{Z}_2$-grading expressed in the vector space decomposition

$$g = g_0 \oplus g_1.$$

Assuming that there exists a homogeneous basis one assigns to elements x in the even subspace g_0 the degree $|x| = 0$ while elements y in the odd subspace g_1 carry the degree $|y| = 1$. The superalgebraic structure is invoked by introducing the superbracket $[\cdot, \cdot] : g \times g \to g$

obeying super antisymmetry and the super Jacobi identity for the homogeneous elements $x, y, z \in g_0$ or g_1,

$$[x, y] + (-1)^{|x||y|}[y, x] = 0$$
$$(-)^{|x||z|}[[x, y], z] + (-)^{|y||x|}[[y, z], x] + (-)^{|z||y|}[[z, x], y] = 0.$$

Similar to the non-graded case, one can classify all contragredient superalgebras (CSA) by Cartan matrices respectively root systems Φ. One might in particular introduce the gradation of the algebra by first grading its root system $\Phi = \Phi_0 \cup \Phi_1$. Let $\langle\cdot,\cdot\rangle$ denote the super Killing form, then one distinguishes the following types of roots:

- if $\langle\alpha|\alpha\rangle \neq 0$ and $2\alpha \notin \Phi$, then $\alpha \in \Phi_0$ is called even or white;
- if $\langle\alpha|\alpha\rangle \neq 0$ and $2\alpha \in \Phi$, then $\alpha \in \Phi_1$ is odd and called black;
- if $\langle\alpha|\alpha\rangle = 0$, then $\alpha \in \Phi_1$ is odd and called grey.

This grading of the root system determines the grading of the algebra by assigning to the step operators $\bar{e}_\alpha, \bar{e}_{-\alpha}$ the same degree as their associated root. Note in particular that one has $\bar{e}_\alpha^2 = \bar{e}_{-\alpha}^2 = 0$ for grey roots. The elements of the Cartan subalgebra are chosen to be even. Instead of working with the whole root system, it is convenient to choose a set of simple roots $\Pi := \{\alpha_1, \dots, \alpha_r\} \subset \Phi$ and to introduce the symmetric Cartan matrix $A_{ij} = \langle\alpha_i|\alpha_j\rangle$ as well as the corresponding Chevalley–Serre basis which determine the superalgebra completely.

Definition 2.1. *Given a root system $\Phi = \Phi_0 \cup \Phi_1$ and a set of simple roots $\Pi \subset \Phi$ with symmetric Cartan matrix $A \in \mathbb{Z}^r \otimes \mathbb{Z}^r$ we assign to it the following unique superalgebra $g = g(A, \Pi)$ whose Chevalley–Serre generators $\{h_i, \bar{e}_i \equiv \bar{e}_{\alpha_i}, \bar{f}_i \equiv \bar{e}_{-\alpha_i}\}_{\alpha_i \in \Pi}$ obey the relations:*

(CSA1).

$$[h_i, h_j] = 0 \qquad [h_i, \bar{e}_j] = A_{ij}\bar{e}_j \qquad [h_i, \bar{f}_j] = -A_{ij}\bar{f}_j \qquad [\bar{e}_i, \bar{f}_j] = \delta_{ij}h_i.$$

(CSA2) Chevalley–Serre relations.

$$(\mathrm{ad}\bar{e}_i)^{1-a_{ij}}\bar{e}_j = (\mathrm{ad}\bar{f}_i)^{1-a_{ij}}\bar{f}_j = 0 \qquad with \quad a_{ij} := \begin{cases} 2A_{ij}/A_{ii} & A_{ii} \neq 0 \\ -1 & A_{ii} = 0 \quad A_{ij} \neq 0 \\ 0 & else. \end{cases}$$

(CSA3). In addition, one has to impose extra Serre relations whenever grey roots are present, i.e. $A_{ii} = \langle\alpha_i|\alpha_i\rangle = 0$ for some $\alpha_i \in \Pi$. For example, provided that

$$A_{ij} = -A_{ik} \neq 0 \quad \text{and} \quad A_{ii} = A_{jk} = 0$$

one has the relation

$$[[\bar{e}_i, \bar{e}_j], [\bar{e}_i, \bar{e}_k]] = [[\bar{f}_i, \bar{f}_j], [\bar{f}_i, \bar{f}_k]] = 0.$$

For a complete list of the extra Serre relations, for which no universal formula is known, we refer the reader to [55].

We emphasize that in the presence of grey roots one encounters two new features which are characteristic of superalgebras and have no analogue for ordinary simple algebras. One of them is the occurrence of the extra Serre relations mentioned in (CSA3). The other one concerns the existence of inequivalent root systems for the same CSA.

If the simple root system of $g(A, \Pi)$ contains grey roots one might generate a different simple root system Π' by applying generalized Weyl reflections associated with grey roots to Π [56, 57],

$$\sigma_i\alpha_j = \alpha_j - a_{ij}\alpha_i. \tag{1}$$

The inequivalent root system might have a different number of odd roots and its associated Cartan matrix A' cannot be related to A by a similarity transformation in general. However, the algebras $g(A, \Pi)$ and $g(A', \Pi')$ generated by the two different Chevalley–Serre bases are isomorphic. There always exists a unique simple root system, called distinguished, where the number of even simple roots is maximal. From this root system, one can construct successively all inequivalent simple root systems, see [56] for details.

Remark 2.1. Henceforth, we shall always work in the distinguished root system and its associated Chevalley–Serre basis.

We now turn to the definition of quantum superalgebras which are constructed as q-deformation of the universal enveloping algebra $U(g)$. The latter is obtained from the graded tensor algebra $\bigoplus_n g^{\otimes n}$ with tensor product

$$(1 \otimes x) \otimes (y \otimes 1) = (-1)^{|x||y|} y \otimes x. \tag{2}$$

On dividing the ideal generated from elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]. \tag{3}$$

In other words, we might identify the superbracket with the supercommutator in $U(g)$. For convenience, we drop from now on the tensor product sign $\otimes$ in $U(g)$. We are now prepared to define the q-deformation of the universal enveloping algebra.

Definition 2.2. *The* quantum universal enveloping superalgebra $U_q(g)$ *is the algebra of power series in the Chevalley–Serre generators* $\{e_i, f_i, h_i\} \cup \{1\}$ *subject to the following supercommutation relations:*

(QSA1). Let A denote the symmetric Cartan matrix associated with the superalgebra g. Then

$$[h_i, h_j] = 0 \quad \text{or equivalently} \quad q^{h_i} q^{-h_i} = q^{-h_i} q^{h_i} = 1 \tag{4}$$

and

$$q^{h_i} e_j q^{-h_i} = q^{A_{ji}} e_j \qquad q^{h_i} f_j q^{-h_i} = q^{-A_{ji}} f_i \qquad [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}. \tag{5}$$

(QSA2). In addition, the generators obey the quantum Serre relations

$$(\mathrm{ad}_{q^{\pm 1}} e_i)^{1-a_{ij}} e_j = 0 \qquad \text{and} \qquad (\mathrm{ad}_{q^{\pm 1}} f_i)^{1-a_{ij}} f_j = 0 \qquad (i \neq j, A_{ii} \neq 0) \tag{6}$$

where the q-deformed adjoint action is defined in terms of the q-deformed supercommutator

$$[e_\alpha, e_\beta]_{q^{\pm 1}} := e_\alpha e_\beta - (-)^{|\alpha||\beta|} q^{\pm\langle\alpha|\beta\rangle} e_\beta e_\alpha. \tag{7}$$

(QSA3). In the case grey roots are present ($A_{ii} = 0$) there are extra quantum Serre relations. For example, under the same conditions as stated in (CSA3) above one has

$$[[e_i, e_j]_{q^{\pm 1}}, [e_i, e_k]_{q^{\pm 1}}]_{q^{\pm 1}} = 0.$$

Similar relations hold for the generators f_i.

The so-defined quantum superalgebras can be endowed with the structure of a Hopf algebra. We choose the following conventions for coproduct and antipode:

$$\begin{aligned} \Delta(q^{h_i}) &= q^{h_i} \otimes q^{h_i} & \gamma(q^{h_i}) &= q^{-h_i} \\ \Delta(e_i) &= e_i \otimes q^{-\frac{h_i}{2}} + q^{\frac{h_i}{2}} \otimes e_i & \gamma(e_i) &= -q^{-\rho} e_i q^{\rho} \\ \Delta(f_i) &= f_i \otimes q^{-\frac{h_i}{2}} + q^{\frac{h_i}{2}} \otimes f_i & \gamma(f_i) &= -q^{-\rho} f_i q^{\rho}. \end{aligned} \tag{8}$$

Here ρ is the unique element in the Cartan subalgebra which satisfies $\alpha_i(\rho) = \langle\alpha_i|\alpha_i\rangle/2$ for any simple root α_i. Note that this defines in fact a graded Hopf algebra, i.e. the coproduct preserves the grading $U_q(g) = U_q(g)_0 \oplus U_q(g)_1$ inherited from $g = g_0 \oplus g_1$,

$$U_q(g)_0 \stackrel{\Delta}{\to} U_q(g)_0 \otimes U_q(g)_0 + U_q(g)_1 \otimes U_q(g)_1 \tag{9}$$

and

$$U_q(g)_1 \stackrel{\Delta}{\to} U_q(g)_0 \otimes U_q(g)_1 + U_q(g)_1 \otimes U_q(g)_0. \tag{10}$$

Moreover, the antipode obeys

$$\gamma(xy) = (-1)^{|x||y|}\gamma(y)\gamma(x). \tag{11}$$

As we infer from the definition of the coproduct the quantum superalgebra $U_q(g)$ is in general non-cocommutative. In formulae, this means that the action of the 'opposite' coproduct

$$\Delta^{\mathrm{op}} \equiv \pi \circ \Delta \tag{12}$$

does not coincide with the action of Δ. Here π denotes the graded permutation operator,

$$\pi(x \otimes y) = (-1)^{|x||y|} y \otimes x. \tag{13}$$

However, both coproduct structures can be related by an invertible element, the universal R-matrix $\mathcal{R} \in U_q(g) \otimes U_q(g)$,

$$\Delta^{\mathrm{op}}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}. \tag{14}$$

In addition, the R-matrix has to satisfy the following well-known identities:

$$\begin{aligned}(1 \otimes \Delta)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12} \\ (\Delta \otimes 1)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} \\ (\gamma \otimes 1)\mathcal{R} &= (1 \otimes \gamma^{-1})\mathcal{R} = \mathcal{R}^{-1}.\end{aligned} \tag{15}$$

From the first two relations and the defining property of the R-matrix, one infers that it provides a constant solution to the graded Yang–Baxter equation in $U_q(g) \otimes U_q(g) \otimes U_q(g)$,

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{16}$$

Here the lower indices indicate on which pair of copies the R-matrix acts. Note that the grading is hidden in the definition of the tensor product (2). In order to obtain spectral parameter dependent solutions one has to consider affine superalgebras $\hat{g}$. The affine extensions can be analogously defined as in the non-graded case and amount to adding one more triplet of generators $\{e_0, f_0, h_0\}$ associated with the affine root α_0 to the Chevalley–Serre basis. For a listing of the possible affine extensions of CSAs in terms of (affine extended) Cartan matrices, respectively Dynkin diagrams, we refer the reader to, e.g., [20, 55, 58]. It is then convenient to introduce the normalized or truncated R-matrix

$$R = q^{-c\otimes d - d\otimes c}\mathcal{R} \tag{17}$$

with c being the central element of the affine superalgebra $\hat{g}$ and d being the homogeneous degree operator defined by the commutation relations

$$[d, e_i] = \delta_{i0}e_i \qquad [d, f_i] = -\delta_{i0}f_i \qquad [d, h_i] = 0 \tag{18}$$

and the Hopf algebra relations

$$\Delta(d) = d \otimes 1 + 1 \otimes d \qquad \gamma(d) = -d. \tag{19}$$

The truncated R-matrix can now be given a spectral parameter dependence by introducing the automorphism

$$D_z(x) = z^d x z^{-d} \qquad z \in \mathbb{C} \quad x \in U_q(\hat{g}). \tag{20}$$

and setting

$$R(z) = (D_z \otimes 1)R = (1 \otimes D_{z^{-1}})R. \tag{21}$$

From the above equation for the universal R-matrix, we now infer that the spectral parameter dependent R-matrix satisfies the equation

$$R_{12}(z)R_{13}(zwq^{1\otimes c\otimes 1})R_{23}(w) = R_{23}(w)R_{13}(zwq^{-1\otimes c\otimes 1})R_{12}(z) \tag{22}$$

which is the known form of statistical mechanics, respectively factorizable S-matrix theory, provided we choose a representation where the central element c is set to zero. The other relations enjoyed by the universal R-matrix translate to

$$\begin{aligned}(\Delta_z \otimes 1)R(w) &= R_{13}(zwq^{1\otimes c\otimes 1})R_{23}(w)\\ (1 \otimes \Delta_z)R(w) &= R_{13}(wz^{-1}q^{-1\otimes c\otimes 1})R_{12}(w)\\ (\gamma \otimes 1)R(z) &= R(zq^{c\otimes 1})^{-1}\\ (1 \otimes \gamma^{-1})R(z) &= R(zq^{-1\otimes c})^{-1}\end{aligned} \tag{23}$$

as well as the quasi-triangular property

$$R(z)\Delta_z(x) = q^{-d\otimes c - c\otimes d}\Delta_z^{\mathrm{op}}(x)q^{d\otimes c + c\otimes d}R(z) \tag{24}$$

which amounts to Jimbo's celebrated equations [7] for the construction of R-matrices. Again we emphasize that we are going to consider only finite-dimensional representations $\rho_V : U_q(\hat{g}) \to \mathrm{End}(V)$ where $c = 0$. Then the above q-factors drop out.

3. The restricted algebra

In this section, we discuss the restricted quantum algebra $U_q^{\mathrm{res}}(\hat{g})$ which will be defined similarly as in the non-graded case (compare [46]). We recall from the non-graded case that the representation theory of $U_q^{\mathrm{res}}(\hat{g})$ as opposed to the non-restricted form $U_q(\hat{g})$ in the root of unity limit is different (see [45–47]). As shown in [14, 18] for non-graded affine algebras, the relevant structure for integrable models is given by $U_q^{\mathrm{res}}(\hat{g})$ whence we define here the analogue of this particular realization for quantum affine superalgebras.

Definition 3.1. *The restricted quantum superalgebra $U_q^{\mathrm{res}}(\hat{g})$ is the algebra generated by the elements*

$$\left\{e_i^{(n)}, f_i^{(n)}, q^{h_i}, q^{-h_i}\right\}_{n\in\mathbb{N}} \qquad with \quad e_i^{(n)} := \frac{e_i^n}{[n]_{q_i}!} \quad f_i^{(n)} := \frac{f_i^n}{[n]_{q_i}!}. \tag{25}$$

Here we have introduced the following super q-integers associated with a simple root α_i:

$$[n]_{q_i} := \frac{(-)^{|i|n}q_i^n - q_i^{-n}}{(-)^{|i|}q_i - q_i^{-1}} = (-)^{|i|(n-1)}[n]_{q_i^{-1}} \qquad q_i := q^{\frac{\langle\alpha_i|\alpha_i\rangle}{2}} \tag{26}$$

and the corresponding factorials as well as binomial coefficients

$$[m]_{q_i}! := \prod_{n=1}^{m}[n]_{q_i} \qquad \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} := \frac{[m]_{q_i}!}{[n]_{q_i}!\,[m-n]_{q_i}!} = (-)^{|i|n(m-n)}\begin{bmatrix} m \\ n \end{bmatrix}_{q_i^{-1}}. \tag{27}$$

To motivate this definition, we recall from the non-graded case that it is crucial to divide the generators by the q-integers in order to obtain a non-trivial root of unity limit, $q^N \to 1$, for the Nth-power of the step operators e_i, f_i which otherwise would become central. Note that for even roots (25) reduces to the non-graded restricted quantum algebra since the super

q-integers (26) turn into ordinary q-integers $\lfloor n \rfloor_q = (q^n - q^{-n})/(q^1 - q^{-1})$. In fact, one might define the super q-integers in terms of the ordinary ones by the following relation:

$$[n]_{q_i} = (\sqrt{-1})^{|i|(n-1)} \lfloor n \rfloor_{\bar{q}_i} \qquad \text{with} \quad \bar{q}_i = (\sqrt{-1})^{|i|} q_i. \tag{28}$$

As an immediate consequence of this relation one has the following lemma.

Lemma 3.1. *The super q-integers satisfy the following three identities:*

$$[n]_{q_i} = q_i^{-n+1} \sum_{l=0}^{n-1} (-)^{|i|l} q_i^{2l} \tag{29}$$

$$[m+n]_{q_i} = (-)^{n|i|} q_i^n [m]_{q_i} + q_i^{-m} [n]_{q_i} \tag{30}$$

$$\prod_{k=0}^{n-1} \left(1 + (-)^{|i|k} q_i^{2k} z\right) = \sum_{l=0}^{n} (-)^{|i|\frac{l(l-1)}{2}} q_i^{l(n-1)} \begin{bmatrix} n \\ l \end{bmatrix}_{q_i} z^l. \tag{31}$$

These formulae together with the definition of the super q-integers will not only simplify the calculations considerably but also allow us to write down compact and elegant formulae, since the additionally introduced sign factors in (26) will turn out to reflect conveniently the grading of the superalgebra.

Having defined the restricted algebra $U_q^{\mathrm{res}}(\hat{g})$ we consider the supercommutator of its generators. One obtains by induction the following formula, whose proof can be found in the appendix:

$$\left[e_i^{(m)}, f_i^{(n)}\right] = \sum_{k=1}^{\min(m,n)} (-)^{|i|(m-k)(n-k)} f_i^{(n-k)} e_i^{(m-k)} \prod_{l=1}^{k} \frac{[h_i; m-n-l+1]}{[l]_{q_i}}. \tag{32}$$

Here we have introduced the Cartan elements

$$[h_i; m] := \frac{q^{h_i} q_i^m - (-)^{|i|m} q^{-h_i} q_i^{-m}}{q - q^{-1}}. \tag{33}$$

These formulae generalize those obtained in [43, 46] to the quantum superalgebra case and will be repeatedly used when we discuss the root of unity limit. We also rewrite the quantum Serre relations. By induction one proves (see appendix) that the m-fold q-deformed adjoint action can be explicitly written as

$$(\mathrm{ad}_{q^{\pm 1}} e_i)^m e_j^n = \sum_{s=0}^{m} (-1)^{s(1+n|i||j|)+|i|\frac{s(s-1)}{2}} q_i^{\mp s(1-na_{ij}-m)} \begin{bmatrix} m \\ s \end{bmatrix}_{q_i^{\pm 1}} e_i^{m-s} e_j^n e_i^s. \tag{34}$$

Setting $m = 1 - a_{ij}, n = 1$ one obtains the quantum Serre relations stated in the definition (QSA2). Using the general formula for arbitrary m, n we now derive the analogue of Lustzig's quantum higher order Serre relations [46] for superalgebras.

Proposition 3.1. *Define the following element in $U_q(\hat{g})$,*

$$\Theta_{m,n} := \frac{(\mathrm{ad}_q e_i)^m e_j^n}{[m]_{q_i}![n]_{q_i}!} = \sum_{r+s=m} (-)^{s+x_s^n} q_i^{-s(1-na_{ij}-m)} e_i^{(r)} e_j^{(n)} e_i^{(s)} \tag{35}$$

with the degree function set to

$$x_s^n := |i| \frac{s(s-1)}{2} + |i||j|ns. \tag{36}$$

Then provided that $m \geqslant 1 - na_{ij} > 0$ *one has*

$$\Theta_{m,n} = 0. \tag{37}$$

We shall refer to these identities as the higher order quantum Serre relations.

Proof. The proof proceeds by induction along the lines for the non-graded case. For $n = 1$ and $m = 1 - na_{ij}$ we obtain the ordinary quantum Serre relations as already seen above. Assuming that the assertion is true for some $n - 1, n > 1$, one performs the induction step by means of the following identities which can be verified by direct calculation.

Setting $m_o := 1 - na_{ij}$ one has

$$\left[\Theta_{m_o,n}, f_i\right] = [1 - (-)^{|i|na_{ij}}]\frac{q^{h_i}}{q - q^{-1}}\Theta_{m_o-1,n} = 0. \tag{38}$$

Note that the vanishing of the supercommutator only follows if $|i| = 0$ or $a_{ij} = 0 \bmod 2$. The first case is trivial. Suppose that $|i| = 1$ then a quick study of the Cartan matrices or the Dynkin diagrams reveals that a_{ij} with α_i being a black simple root is satisfied for all Kac–Moody superalgebras.

The second identity we exploit is given by

$$\left[\Theta_{m_o,n}, f_j\right] = (-)^{|j|(n-1)}\left\{\frac{q^{-h_j}q_j^{-(1-m_o a_{ji}-n)}}{q^{-1} - q}\Theta_{m_o,n-1}\right\} + (-)^{|j|\frac{n(n-1)}{2}}\{q \to q^{-1}\} \tag{39}$$

where the last term in brackets is obtained from the first one when replacing q by q^{-1}. Also this supercommutator vanishes by the induction hypothesis.

Thus, we conclude that $\Theta_{m_o,n}$ supercommutes with f_i, f_j and therefore trivially with all $f_k, k = 0, 1, \ldots, r$. Hence, it must vanish. This completes the induction proof for $m = m_o$. Using the defining relation of $\Theta_{m,n}$ we see that the assertion also follows for larger m. □

For future purposes, it will be useful to rewrite these higher order quantum Serre relations in a different form.

Corollary 3.1. *For* $m \geqslant 1 - na_{ij} > 0$ $(|i| = 0$ *or* $a_{ij} = 0 \bmod 2)$ *the above relations can be rewritten as*

$$0 = e_i^{(m)}e_j^{(n)} + \sum_{\substack{r+s=m \\ m+na_{ij}\leqslant s\leqslant m}} c_s e_i^{(r)}e_j^{(n)}e_i^{(s)} \tag{40}$$

with the coefficient function equal to

$$c_s := (-)^{s+x_s^n}q_i^{-s(1-na_{ij}-m)}\sum_{p=0}^{m+na_{ij}-1}(-)^{p+|i|\frac{p(p-1)}{2}}q_i^{-p(s-1)}\begin{bmatrix} s \\ p \end{bmatrix}_{q_i}. \tag{41}$$

Here the degree function x_s^n *is defined as in (36).*

Proof. First, one observes that for $0 \leqslant p \leqslant m + na_{ij} - 1$ one has

$$0 = \sum_{p=0}^{m+na_{ij}-1}(-)^{|i||j|np}q_i^{p(na_{ij}+m-p)}\Theta_{m-p,n}e_i^{(p)}.$$

Plugging in the definition of $\Theta_{m,n}$ and exploiting formula (31) for $z = -1$ the assertion follows. □

Since we will have to consider below higher tensor products of the restricted algebra in order to make contact with integrable lattice models we conclude this section by stating the coproduct formulae for $U_q^{\mathrm{res}}(\hat{g})$. Taking into account that coproduct (8) is a Hopf algebra homomorphism one obtains by induction

$$\Delta\left(e_i^{(m)}\right) = \sum_{n=0}^{m} e_i^{(n)} q^{(m-n)\frac{h_i}{2}} \otimes e_i^{(m-n)} q^{-n\frac{h_i}{2}}. \tag{42}$$

A similar formula holds for the generators $f_i^{(n)}$. At first sight, this formula does not seem to differ from the non-graded case. However, we point out that the graded tensor product (2) has to be taken into account and that its structure is conveniently hidden in the definition of the super q-integers (26). In order to construct higher tensor products, one defines iteratively the L-fold coproduct via

$$\Delta^{(L)} = (\Delta \otimes 1)\, \Delta^{(L-1)} \qquad \text{with} \quad \Delta^{(2)} \equiv \Delta. \tag{43}$$

Taking formula (42) as a starting point and exploiting once more the fact that the coproduct is an algebra homomorphism, one arrives at

$$\Delta^{(L)}\left(e_i^{(m)}\right) = \sum_{0=n_0\leqslant\cdots\leqslant n_L=m} \bigotimes_{l=1}^{L} e_i^{(n_l-n_{l-1})} q^{(m-n_l-n_{l-1})\frac{h_i}{2}}. \tag{44}$$

Analogous formulae hold for the generators f_i. Again, we stress that this formula resembles closely that in the non-graded case since the grading is conveniently encoded in the definition of the super q-integers.

4. The root of unity limit of $U_q^{\mathrm{res}}(\hat{g})$

In this section, we discuss the root of unity limit $q^N \to 1$ and henceforth we assume that $q^N = 1$ is primitive. Since the outcome will depend crucially on the grading of the generators we discuss first the restricted subalgebra $U_q^{\mathrm{res}}(\hat{g})_i$ associated with a single simple root α_i which is either white, black or grey. $U_q^{\mathrm{res}}(\hat{g})_i$ is generated by the elements

$$\left\{e_i^{(n)}, f_i^{(n)}, q^{h_i}, q^{-h_i}\right\}_{n\in\mathbb{N}} \qquad \text{with} \quad e_i^{(n)} := \frac{e_i^n}{[n]_{q_i}!} \quad f_i^{(n)} := \frac{f_i^n}{[n]_{q_i}!}. \tag{45}$$

As mentioned in the previous section, it is important to divide the generators by the super q-integers. From the non-graded case, it is known that the elements $e_i^{N'}$, $f_i^{N'}$, $q^{\pm h_i}$ (α_i white) become central in the root of unity limit $q^N \to 1$ and in non-cyclic representations take the values $e_i^{N'}, f_i^{N'} = 0$, $q^{\pm h_i} = \pm 1$ [45], where the integer N' is defined as

$$N' := \begin{cases} N & N \text{ odd} \\ N/2 & N \text{ even.} \end{cases} \tag{46}$$

In contrast, the restricted generators $e_i^{(N')}$, $f_i^{(N')}$ stay well defined due to a simultaneous vanishing of the q-integer $[N']_{q_i} = 0$ [46]. We will now consider the analogue of these non-cyclic representations for arbitrary roots α_i and shall impose the preliminary commensurability condition

$$\lambda(h_i) = 0 \bmod N' \quad \Leftrightarrow \quad q^{h_i} = \pm 1 \tag{47}$$

with λ denoting the highest weight determining the representation. The above commensurability condition is preliminary, since we will have to strengthen it later on when discussing translation invariance of the restricted algebra for higher tensor products.

We are interested in the supercommutation relations of the restricted generators at roots of unity. From the general formula (32) of the previous section, we see that for $m = n = N'$

the product in the supercommutator always contains a zero for each summand in the root of unity limit. Therefore, its expression simplifies to

$$\lim_{q^N\to 1}\left[e_i^{(N')}, f_i^{(N')}\right] = \lim_{q^N\to 1}\frac{[h_i;0]}{[N']_{q_i}}\prod_{l=1}^{N'-1}\frac{[h_i;-l]}{[l]_{q_i}}$$
$$= \left(\frac{(-)^{|i|}q_i - q_i^{-1}}{q-q^{-1}}\right)^{N'}\prod_{l=1}^{N'-1}\frac{q^{h_i}q_i^{-l}-(-)^{|i|l}q^{-h_i}q_i^l}{(-)^{|i|l}q_i^l-q_i^{-l}}\lim_{q^N\to 1}\frac{q^{h_i}-q^{-h_i}}{(-)^{|i|N'}q_i^{N'}-q_i^{-N'}}.$$

Note that the supercommutator is only non-zero if the super q-integer $[N']_{q_i}$ vanishes. We will now evaluate the limit in the second line of the above equation for the different cases of α_i being a white, black or grey root.

α_i white. In this case, all supercommutators and super q-integers reduce to ordinary commutators and q-integers respectively. The discussion of the root of unity limit follows along the lines in [18]. Setting $\langle\alpha_i|\alpha_i\rangle = 2$ one finds that

$$\lim_{q^N\to 1}\left[e_i^{(N')}, f_i^{(N')}\right] = (-)^{N'-1}q^{N'h_i}\frac{h_i}{N'}. \tag{48}$$

Together with the obvious commutation relations

$$\left[h_i, e_i^{(N')}\right] = 2N'e_i^{(N')} \quad \text{and} \quad \left[h_i, f_i^{(N')}\right] = -2N'f_i^{(N')} \tag{49}$$

we conclude that for every white root we obtain a non-deformed $U(sl_2)$ algebra in the root of unity limit generated by $\left\{e_i^{(N')}, f_i^{(N')}, h_i/N'\right\}$. Note that for odd roots of unity we always have the real form $sl_2(\mathbb{R})$, while for even roots of unity one might also obtain $sl_2(\mathbb{C})$.

α_i black. For black roots, we infer from the definition of the super q-integers that for odd roots of unity $q^N = 1$ the q-integer $[N]_{q_i}$ does not necessarily vanish. Let us therefore first consider the case of taking an even primitive root of unity $q^N = 1$, with N' even $\left(q = \mathrm{e}^{\mathrm{i}\pi\frac{2k-1}{N'}}, k\in\mathbb{N}\right)$. Then obviously $q^{\frac{N'}{2}} = \pm\sqrt{-1}$ and

$$[N'/2]_{q_i} = \frac{q^{\frac{N'}{2}} + q^{-\frac{N'}{2}}}{q^1+q^{-1}} = 0. \tag{50}$$

Considering the sector $\lambda(h_i) = 0 \bmod N'$ one arrives at the supercommutator

$$\lim_{q^N\to 1}\left[e_i^{(\frac{N'}{2})}, f_i^{(\frac{N'}{2})}\right] = q^{\frac{N'}{2}}q^{-h_i}\left(\frac{q^1+q^{-1}}{q^1-q^{-1}}\right)^{\frac{N'}{2}}\frac{2h_i}{N'}. \tag{51}$$

Moreover, the supercommutation relations with h_i analogous to the white case hold,

$$\left[h_i, e_i^{(\frac{N'}{2})}\right] = N'e_i^{(\frac{N'}{2})} \quad \text{and} \quad \left[h_i, f_i^{(\frac{N'}{2})}\right] = -N'f_i^{(\frac{N'}{2})}. \tag{52}$$

After a suitable renormalization of the step operators, these supercommutation relations resemble closely those of an (non-deformed) $osp(2|1)$ subalgebra. However, the factor q^{-h_i} in (51) produces in general alternating signs in a multiplet as can immediately be seen by means of the above commutation relations with the step operators.

Therefore, we consider instead the even generators $e_i^{(N')}, f_i^{(N')}$ which obey

$$\lim_{q^N\to 1}\left[e_i^{(N')}, f_i^{(N')}\right] = -\left(\frac{q^1+q^{-1}}{q^1-q^{-1}}\right)^{N'}q^{(N'-1)h_i}\lim_{q^N\to 1}\frac{q^{h_i}-q^{-h_i}}{q^{N'}-q^{-N'}}$$
$$= -\left(\frac{q^1+q^{-1}}{q^1-q^{-1}}\right)^{N'}q^{N'h_i}\frac{h_i}{N'} \tag{53}$$

and

$$\left[h_i, e_i^{(N')}\right] = 2N' e_i^{(N')} \quad \text{and} \quad \left[h_i, f_i^{(N')}\right] = -2N' f_i^{(N')}. \tag{54}$$

From the last commutation relations, one deduces that the factor in front of h_i/N' in equation (53) always stays positive. Hence, the bosonic generators associated with a black root yield after a suitable renormalization again an $sl_2(\mathbb{R})$ algebra.

Analogously, we might consider for roots of unity $q^N = 1$ with N' even or odd the even generators $e^{(2N')}$, $f^{(2N')}$ which satisfy the commutation relations

$$\begin{aligned}\lim_{q^N\to 1}\left[e_i^{(2N')}, f_i^{(2N')}\right] &= -q^{h_i}\left(\frac{q^1+q^{-1}}{q^1-q^{-1}}\right)^{2N'} \lim_{q^N\to 1}\frac{q^{h_i}-q^{-h_i}}{q^{2N'}-q^{-2N'}} \\ &= -\left(\frac{q^1+q^{-1}}{q^1-q^{-1}}\right)^{2N'} \frac{h_i}{2N'}\end{aligned} \tag{55}$$

and

$$\left[h_i, e_i^{(2N')}\right] = 4N' e_i^{(2N')} \quad \text{and} \quad \left[h_i, f_i^{(2N')}\right] = -4N' f_i^{(2N')}. \tag{56}$$

After a suitable renormalization of the step operators we obtain once more an $sl_2(\mathbb{R})$ subalgebra.

α_i grey. In this case, the generators are idempotent, $e_i^2, f_i^2 = 0$ and the above considerations cannot be applied.

4.1. The quantum Frobenius homomorphism

Having analysed the subalgebras associated with a single simple root α_i in the limit $q^N \to 1$, we now address the remaining algebraic structures, namely the supercommutation and Serre relations involving generators of different roots. We recall from the non-graded case that the relations of the restricted algebra $U_q^{\text{res}}(\hat{g})$ at roots of unity can be partially identified with the relations of the non-deformed algebra $U(\hat{g})$ via Lustzig's quantum Frobenius homomorphism.

From the results of our case-by-case discussion of $U_q^{\text{res}}(\hat{g})_i$ with α_i being either white, black or grey we anticipate that for affine superalgebras we will restrict ourselves at least to the even (or bosonic) subalgebra in order to give a well-defined analogue of the quantum Frobenius mapping for the graded case [46].

Theorem 4.1. *Let $\hat{g}$ be a affine superalgebra and denote by $\hat{g}^{\text{trunc}}$ the algebra which is obtained by deleting from the Dynkin diagram of $\hat{g}$ the nodes (and their adjoint edges) corresponding to grey roots. Then at roots of unity $q^N = 1$ with N odd the mapping $F : U_q^{\text{res}}(\hat{g}) \to U\left(\hat{g}_0^{\text{trunc}}\right)$ defined by $F(q^{h_i}) = 1$ and*

$$F\left(e_{\pm\alpha_i}^{(m)}\right) = \begin{cases} \bar{e}_{\pm\alpha_i}^{m/N}\big/(m/N)! & m = 0 \bmod N \quad \alpha_i \textit{ white} \\ \bar{e}_{\pm\alpha_i}^{m/2N}\big/(m/2N)! & m = 0 \bmod 2N \quad \alpha_i \textit{ black} \quad \left(e_{-\alpha_i} \equiv f_i\right) \\ 0 & \textit{else} \end{cases} \tag{57}$$

is a Hopf algebra homomorphism. Here $\hat{g}_0^{\text{trunc}}$ denotes the even subalgebra of $\hat{g}^{\text{trunc}}$. For examples, see table 1. Note in particular that provided there are no grey roots one has an homomorphism $U_q^{\text{res}}(\hat{g}) \to U(\hat{g}_0)$ with $\hat{g}_0$ being the even subalgebra of $\hat{g}$.

Proof. For the subalgebras generated by the Chevalley–Serre basis associated with white simple roots, we can apply the results of the non-graded case [18]. Recall that in general the

Table 1. Listed are the various examples of Kac–Moody superalgebras and the associated truncated algebra obtained by deleting the grey nodes in the Dynkin diagram.

$\hat{g}$	$\hat{g}^{\text{trunc}}$	$\hat{g}_0^{\text{trunc}}$
$osp(2m\|2n)$	$sl_n \oplus so_{2m+1}$	$sl_n \oplus so_{2m+1}$
$sl(2n\|2n)$	$sl_n \oplus sl_n$	$sl_n \oplus sl_n$
$G(3)$	G_2	G_2
$osp(2\|2n)$	$osp(2\|2n)$	sp_{2n}
$osp(2m\|2n)^{(1)}$	$sp_{2n} \oplus so_{2m+1}$	$sp_{2n} \oplus so_{2m+1}$
$osp(2\|2)^{(2)}$	$osp(2\|2)^{(2)}$	$sl_2^{(1)}$
$osp(2\|2n)^{(1)}$	$osp(2\|2n)^{(1)}$	$sp_{2n}^{(1)}$

correct Chevalley–Serre relations have only been obtained for odd roots of unity in this case, whence we restrict ourselves to N odd. When dealing with the bosonic generators $e_i^{(2N)}$, $f_i^{(2N)}$ obtained from a black root α_i we first of all immediately verify that

$$\left[h_i, e_j^{(N)}\right] = NA_{ij}e_j^{(N')} \qquad \text{and} \qquad \left[h_j, e_i^{(2N)}\right] = 2NA_{ij}e_j^{(N')} \quad (\alpha_j \text{ white})$$
$$\left[h_i, e_j^{(2N)}\right] = 2NA_{ij}e_j^{(2N)} \qquad \text{and} \qquad \left[h_j, e_i^{(2N)}\right] = 2NA_{ij}e_j^{(N)} \quad (\alpha_j \text{ black}).$$

The second case only occurs for $\hat{g} = osp(2|2)^{(2)}$. It remains to show that the Serre relations in $U(\hat{g}_0)$ are satisfied. For this purpose, we will exploit the higher quantum Serre relations for superalgebras proved in the previous section.

Starting from the variant (40) one infers that only the cases with α_i black need to be considered. For α_i white, the proof follows along the lines in the non-graded case [18]. Let us start with $|i| = 1$, $|j| = 0$ and set $m = 2N\left(1 - \frac{a_{ij}}{2}\right)$ and $n = N$. Remember that for black roots α_i one always has $a_{ij} = 0 \bmod 2$. Exploiting the algebraic properties of the super q-integers and taking into account that $q^N = 1$ one derives the following limit of the coefficient function (41):

$$\lim_{q^N \to 1} c_s = \begin{cases} (-)^{\frac{s}{2N}} & s = 0 \bmod 2N \\ 0 & \text{else.} \end{cases}$$

Taking this result together with the identity

$$\lim_{q^N \to 1} \frac{[2N]_{q_i}!^s}{[2Ns]_{q_i}!} = \frac{1}{s!}$$

we obtain from the higher order quantum Serre relations of $U_q^{\text{res}}(\hat{g})$ the non-deformed Serre relations

$$0 = \sum_{r+s=1-\frac{a_{ij}}{2}} (-)^s \binom{1 - \frac{a_{ij}}{2}}{s} e_i^{(2N)r} e_j^{(N)} e_i^{(2N)s}.$$

Again, we remind the reader that $a_{ij}/2$ is an integer for all superalgebras provided i labels a black and j a black or white root. For example, if $\hat{g} = osp(2|2n)$ one has $a_{nn-1} = 2$ and the above identity with $a_{nn-1} \to a_{nn-1}/2$ derived in the root of unity limit then just corresponds to the Serre relations of $U(\hat{g}_0 = sp_{2n})$. Note that we have defined the Chevalley–Serre basis in terms of the symmetric Cartan matrix.

Similarly, one obtains for the remaining case that $|i| = |j| = 1$ ($\hat{g} = osp(2|2)^{(2)}$) by setting $m = 2N(1 - a_{ij})$, $n = 2N$ from (40) the relations

$$0 = \sum_{r+s=1-a_{ij}} (-)^s \binom{1 - a_{ij}}{s} e_i^{(2N)r} e_j^{(2N)} e_i^{(2N)s}.$$

As one easily verifies these are also the correct Serre relations of $\hat{g}_0 = sl_2^{(1)}$. We point out that in this special case the theorem might be extended to even roots of unity with N' odd as well, since $osp(2|2)^{(2)}$ contains only black simple roots.

That the homomorphism is also compatible with the coproduct structures can be easily seen from formula (42). This completes the proof. □

Remark 4.1. Note that for generic q the even subalgebra is modified under the q-deformation. For example, while for the non-deformed algebra $U(\hat{g})$ $(q = 1)$ the squared Chevalley–Serre generators $\bar{e}_i^2$, $\bar{f}_i^2$ with α_i black give rise to an sl_2 subalgebra, it is not true that the q-deformed generators e_i^2, f_i^2 of $U_q(\hat{g})$ obey the $U_q(sl_2)$ commutation relations. As we have shown in the previous section and in the above theorem one recovers some of the 'classical' relations in the root of unity limit from the restricted quantum superalgebra $U_q^{\mathrm{res}}(\hat{g})$.

One might wonder if the restriction to consider only the truncated affine superalgebra $\hat{g}^{\mathrm{trunc}}$ might be lifted and the above theorem can be extended to the whole superalgebra as it is the case when grey roots are absent. In order to settle this issue, one would need to have an analogue of the Cartan–Weyl basis for the quantum case. This would allow us to investigate whether there are additional bosonic generators obtained from multiple (q-deformed) supercommutators involving grey step operators. While such a quantum Cartan–Weyl basis has been proposed in the literature [29, 30] it is not clear that it will give rise to the correct root space structure in the root of unity limit. Moreover, since the Cartan–Weyl generators are defined in terms of multiple (q-deformed) supercommutators their coproduct structure is quite intricate. For our physical application in the subsequent section, however, we have to consider higher tensor products. Since we are primarily interested in the physical consequences we leave the issue of a possible extension of the above theorem to include also grey roots to future work.

5. Translation invariance and symmetries of integrable lattice models

In this section, we turn to the physical application of our previous discussion. We shortly review how to each quantum affine superalgebra a lattice model can be assigned and then demonstrate which of the restricted subalgebras treated before are translation invariant and form a symmetry algebra.

Suppose we are given a finite-dimensional representation $\rho_V : U_q(\hat{g}) \to \mathrm{End}(V)$ of the quantum affine superalgebra $U_q(\hat{g})$ over some graded vector space $V = V^{(0)} \oplus V^{(1)}$. Taking an $L \times M$ square lattice we assign to each link of the lattice a copy of the representation space V and to each vertex the spectral parameter dependent R-matrix evaluated in this representation, $R^{VV}(z) = (\rho_V \otimes \rho_V)R(z) \in \mathrm{End}(V \otimes V)$. By abuse of notation we will refer to $R^{VV}(z)$ henceforth simply by $R(z)$ in order to unburden the formulae.

In addition, we will restrict ourselves to degree zero R-matrices. Recall that according to the computational rules of supervector spaces operators carry a degree and are represented by supermatrices whose entries are in general Grassmann numbers. For the integrable models studied in the literature, one usually assumes that the corresponding R-matrix is of degree zero (see e.g. [19])

$$\deg R(z) := |a| + |b| + |c| + |d| + \deg R(z)_{ab}^{cd} = 0 \tag{58}$$

and that the non-vanishing matrix elements $R(z)_{ab}^{cd}$ viewed as Grassmann numbers are also of even degree, $\deg R(z)_{ab}^{cd} = 0$. Here the indices $a, b, c, d = 1, \ldots, \dim V$ refer to some homogeneous basis $\{v_a\} \subset V$ and $|a| = 0, 1$ denotes the degree of the basis vector v_a. In fact, one wants $R(z)_{ab}^{cd}$ to be ordinary positive numbers which can be interpreted as Boltzmann weights. Then there exists a well-defined statistical lattice model whose partition function can be written as a supertrace over the L-fold tensor product

$$Z(w) = \mathrm{str}_{V^{\otimes L}} T(w)^M. \tag{59}$$

Here $T(z)$ denotes the graded transfer matrix which is defined as the partial supertrace of the following operator product:

$$T(z) = \mathrm{str}_{V_0} R_{0L}(z) R_{0L-1}(z) \cdots R_{02}(z) R_{01}(z) \in \mathrm{End}(V_1 \otimes V_2 \otimes \cdots \otimes V_L). \tag{60}$$

The 'auxiliary space' $V_0 \cong V$ labels the boundary values and the remaining spaces $V_1 \otimes V_2 \otimes \cdots \otimes V_L$, $V_i \cong V$ form one row of the lattice. The lower indices indicate on which copy of the representation space the R-matrix acts. It needs to be emphasized that by the choice (60) of the transfer matrix we have obviously imposed periodic boundary conditions leading to translation invariance. Assuming as usual regularity of the R-matrix

$$R(z = 1) = \pi$$

with π being the previously introduced graded permutation operator (13) one obtains from the transfer matrix at $z = 1$ the translation operator

$$T(z = 1) = \mathrm{str}_0 R_{0L}(1) R_{0L-1}(1) \cdots R_{01}(1) = \pi_{0L} \cdots \pi_{02}\pi_{01} =: \Pi^{-1}. \tag{61}$$

The action of Π amounts to the following shift in one row of the lattice:

$$\Pi : V_1 \otimes V_2 \otimes \cdots \otimes V_L \to V_2 \otimes V_3 \otimes \cdots \otimes V_L \otimes V_1. \tag{62}$$

That the transfer matrix commutes with the translation operator might be directly verified from its definition or from the more general formula

$$[T(z), T(w)] = 0. \tag{63}$$

That the transfer matrices of different spectral parameters commute is a direct consequence of the graded Yang–Baxter equation (22) and manifests the integrability of the model.

Besides the transfer matrix, one is often also interested in the spectrum of the formally associated spin-chain Hamiltonian which is defined as[3]

$$H = z\frac{\mathrm{d}}{\mathrm{d}z} \ln T(z)\bigg|_{z=1} = \sum_{n=1}^{L} \pi_{nn+1}\, z\frac{\mathrm{d}}{\mathrm{d}z} R_{nn+1}(z)\bigg|_{z=1} \qquad L+1 \equiv 1. \tag{64}$$

Here π_{nn+1} denotes the graded permutation operator, acting on the nth and $(n+1)$th factors in the spin chain. In the second equation, we have once more exploited the regularity property. From this expression or the commutation relation, one immediately infers that also the Hamiltonian is translation invariant $[H, \Pi] = 0$.

In contrast, the action of the quantum affine superalgebra $U_q(\hat{g})$ on the spin-chain $V_1 \otimes V_2 \otimes \cdots \otimes V_L$ is in general not translation invariant as can be directly seen from the explicit expression of its generators in the L-fold tensor product

$$\begin{aligned}
E_i &= \sum_{n=1}^{L} E_{i;n} := \Delta^{(L)}(e_i) = \sum_{n=1}^{L} q^{\frac{h_i}{2}} \otimes \cdots \otimes q^{\frac{h_i}{2}} \otimes e_i \otimes q^{-\frac{h_i}{2}} \otimes \cdots \otimes q^{-\frac{h_i}{2}} \\
F_i &= \sum_{n=1}^{L} F_{i;n} := \Delta^{(L)}(f_i) = \sum_{n=1}^{L} q^{\frac{h_i}{2}} \otimes \cdots \otimes q^{\frac{h_i}{2}} \otimes f_i \otimes q^{-\frac{h_i}{2}} \otimes \cdots \otimes q^{-\frac{h_i}{2}} \\
q^{H_i} &= \prod_{n=1}^{L} q^{H_{i;n}} := \Delta^{(L)}(q^{h_i}) = q^{h_i} \otimes \cdots \otimes q^{h_i}.
\end{aligned} \tag{65}$$

[3] Note that whether the spin-chain Hamiltonian is Hermitian or not might depend on the value of the deformation parameter and the chosen representation V.

We stress also that these generators (except for the Cartan elements q^{H_i}) for generic values of the deformation parameter q commute neither with the transfer matrix (60) nor with the Hamiltonian (64) due to the periodic boundary conditions.

5.1. Translation invariance at roots of unity

We now demonstrate that at roots of unity certain subalgebras of $U_q^{\mathrm{res}}(\hat{g})$ as discussed in the preceding section are translation invariant. For this purpose, we state first for generic q the transformation law for the restricted generators $E_i^{(m)}$, $m \in \mathbb{N}$,

$$\Pi E_i^{(m)}\Pi^{-1} = E_i^{(m)}q^{mH_{i;L}} + \sum_{n=1}^{m}(-)^{|i|\frac{n(n-1)}{2}}q_i^{n(m-1)}E_i^{(m-n)}E_{i;L}^{(n)}q^{mH_{i;L}}\prod_{l=0}^{n-1}\left((-)^{|i|(m+l+1)}q_i^{-2l}q^{-H_i}-1\right). \tag{66}$$

The proof proceeds by induction and is detailed in the appendix. An analogous formula holds for $F_i^{(m)}$. Letting the deformation parameter now approach a (primitive) root of unity, $q^N \to 1$, and setting $m = N'/2, N', 2N'$ we discuss translation invariance of the Chevalley–Serre step operators treating as before the cases of α_i being even or odd separately.

α_i white. For white roots ($|i| = 0$) the same considerations as in [18] apply and we find that the sl_2 subalgebra commutes or anticommutes with Π depending on whether h_i takes on even or odd integer values in the chosen highest weight representation V of the spin chain (see [18] for details). Explicitly, the $l = 0$ term in the product of formula (66) always vanishes provided that $q^{H_i} = 1$ and we obtain

$$\Pi E_i^{(N')}\Pi^{-1} = E_i^{(N')}q^{N'H_{i;L}}. \tag{67}$$

The commensurability condition (47) has therefore for even roots of unity to be strengthened to $\lambda(H_i) = 0 \bmod N$. Furthermore, we conclude that if $q^{N'H_{i;L}} = 1$ or -1 the generator $E_i^{(N')}$ commutes or anticommutes with the translation operator.

α_i black. For black roots ($|i| = 1$) we discuss the case of even roots of unity with N' even first. Then the generator $E_i^{(N'/2)}$ is odd and the sign factor in the product of (66) vanishes. As before, we find $\Pi E_i^{(\frac{N'}{2})}\Pi^{-1} = E_i^{(\frac{N'}{2})}q^{\frac{N'}{2}H_{i;L}}$ provided that $q^{H_i} = 1$. However, unlike in the white root case the sign of q^{H_i} alternates in a given multiplet due to the supercommutation relations (52). Therefore, the odd generators of black roots are in general not translation invariant.

The situation looks better for the even generators $E_i^{(N')}$, $F_i^{(N')}$. Now the sign factor $(-)^{|i|(N'+l+1)}$ in the product of (66) does not vanish and we have to change the commensurability condition such that $q^{H_i} = -1$ in order to compensate it. Due to the supercommutation relations (54) we see that under the action of the step operators the sign of q^{H_i} stays constant in a given multiplet and from (66) we thus conclude that for $\lambda(H_i/N') \in 2\mathbb{Z}+1$ one has

$$\Pi E_i^{(N')}\Pi^{-1} = E_i^{(N')}q^{N'H_{i;L}}. \tag{68}$$

As in the case of white generators, the value of the remaining factor $q^{N'H_{i;L}}$ depends on the chosen representation V which the spin-chain is built out of. If the Cartan generator h_i takes on even integer values in V then the generators commute with the translation operator; if it is odd integer valued they anticommute.

Table 2. Shown are the various step operators which have a non-trivial root of unity limit and their invariance properties w.r.t. the adjoint action of the translation operator.

α	N	Commensurate sector	Generators	Invariant
White	Odd/even	$\lambda(H_i) = 0 \bmod N$	$E_i^{(N')}, F_i^{(N')}$	Yes
Black	N' even	$\lambda(H_i) = 0 \bmod N'$	$E_i^{(\frac{N'}{2})}, F_i^{(\frac{N'}{2})}$	No
		$\lambda(H_i)/N' \in 2\mathbb{Z}+1$	$E_i^{(N')}, F_i^{(N')}$	Yes
	N' odd	$\lambda(H_i)/N' \in 2\mathbb{Z}+1$	$E_i^{(2N')}, F_i^{(2N')}$	Yes
	Odd	$\lambda(H_i) = 0 \bmod N$	$E_i^{(2N)}, F_i^{(2N)}$	No
Grey	Odd, even	—	E_i, F_i	No

Similarly, one has for even roots of unity $q^N = 1$ with N' being odd that in the sector $q^{H_i} = -1, \lambda(H_i/N') \in 2\mathbb{Z}+1$ the even generator $E_i^{(N)}$ satisfies

$$\Pi E_i^{(N)} \Pi^{-1} = E_i^{(N)}.$$

Note that there is this time no additional factor since the generator is taken to the power $N = 2N'$. We point out that at odd roots of unity there is no obvious way to make the generator $E_i^{(2N)}$ translation invariant, since the desired property $q^{H_i} = -1$ cannot be satisfied here.

α_i grey. Since in this case the step operators are nilpotent, the only possibility is that the generators E_i, F_i are translation invariant. However, this is not the case. We summarize the results of this section in table 2.

Since according to our analysis the step operators $E_i^{(2N')}, F_i^{(2N')}$ associated with a black simple root are only translation invariant at even roots of unity due to the grading, we might in general not expect that the whole algebraic structure as obtained by the quantum Frobenius homomorphism of the previous section is compatible with periodic boundary conditions. Nevertheless, there are several cases where the entire even subalgebra $\hat{g}_0$ is translation invariant (at even roots of unity), e.g. $\hat{g} = osp(2|1)^{(1)}$, $osp(2|2)^{(2)}$ or $sl(1|3)^{(4)}$.

5.2. Boost operator

After having established the analogous results for the restricted quantum superalgebra $U_q^{\text{res}}(\hat{g})$ as in the non-graded case, we are now prepared to apply the analogous line of argument as presented in [18] to show that the various translation invariant subalgebras listed above constitute symmetries of the vertex model. For Hamiltonian (64) one shows directly by exploiting the quantum algebra invariance of R-matrix (24) that it commutes with the respective even subalgebras. In order to show the invariance of the transfer matrix, we act with $w\frac{\mathrm{d}}{\mathrm{d}w}$ on the graded Yang–Baxter equation

$$R_{0j}(z)R_{0j+1}(w)R_{jj+1}(w/z) = R_{jj+1}(w/z)R_{0j+1}(w)R_{0j}(z)$$

and set $z = w$ afterwards to find

$$[\pi_{jj+1}R'_{jj+1}(1), R_{0j+1}(z)R_{0j}(z)] = R_{0j+1}(z)R'_{0j}(z) - R'_{0j+1}(z)R_{0j}(z).$$

Here the prime indicates acting with $z\frac{\mathrm{d}}{\mathrm{d}z}$ on the respective R-matrix and we have employed the regularity property $R(1) = \pi$. From this relation together with translation invariance of the transfer matrix, one finds that the boost operator [59–62] defined as

$$K = \sum_{j \bmod L} j\pi_{jj+1}\, z\frac{\mathrm{d}}{\mathrm{d}z}R_{jj+1}(z)\bigg|_{z=1} \qquad (69)$$

satisfies the crucial relation

$$[K, T(z)] = z\frac{\mathrm{d}}{\mathrm{d}z}T(z). \tag{70}$$

Note that the sum is taken over the integers modulo L and that the R-matrix is of even degree. Upon integration, the last relation implies that under the adjoint action of the boost operator the transfer matrix is shifted in the spectral parameter,

$$w^K T(z) w^{-K} = T(zw). \tag{71}$$

Exploiting now repeatedly translation invariance of the generators listed in table 2, we find first that the boundary terms

$$\pi_{L1}\ z\frac{\mathrm{d}}{\mathrm{d}z}R_{L1}(z)\bigg|_{z=1} \tag{72}$$

of the boost operator commute with the respective subalgebras under the stated commensurability condition. The remaining terms commute also due to quantum algebra invariance of R-matrix (24) and we conclude that the boost operator is invariant under the action of the respective subalgebras. Exploiting translation invariance for the second time, we see from relation (71) that the transfer matrix commutes with the translation invariant generators listed in table 2.

For each white and black root, we have therefore a $U(sl_2)$ invariance of the statistical model in the various commensurate sectors. These different subalgebras can combine to a larger symmetry algebra. For example, when $\hat{g} = sl(2m|2n)^{(1)}$ we have as symmetry algebra $sl_m \oplus sl_n \oplus u(1)$ at odd roots of unity. The associated class of integrable lattice theories includes among other the Perk–Schultz models [34] and the U-model [36, 38]. If the order of the root of unity is even we mention once more the $osp(2|2)^{(2)}$ example [63] whose symmetry algebra is $sl_2^{(1)}$.

6. Conclusions

In this paper, we have introduced the restricted quantum affine superalgebra $U_q^{\mathrm{res}}(\hat{g})$ and investigated its properties in the root of unity limit. We have proved several new identities (for arbitrary values of the deformation parameter q) such as the supercommutator (32) or the higher order Serre relations for quantum affine algebras (40). In the root of unity limit $q^N \to 1$ we then employed these new formulae to prove an analogue of Lustzig's quantum Frobenius homomorphism for the super case. While the odd or fermionic part of the superalgebra does not give rise to interesting structures in this limit, we showed that for the even part $U_q^{\mathrm{res}}(\hat{g})_0$ one recovers the 'classical' algebraic relations. Namely, for superalgebras whose distinguished simple root system contains white and black roots only one obtains the entire even non-deformed subalgebra $U(\hat{g}_0)$ as $q^N \to 1$ with N being odd. When grey roots are present we had to restrict ourselves to a subalgebra $\hat{g}_0^{\mathrm{trunc}} \subseteq \hat{g}_0$. As we pointed out in the text a further extension of the theorem might be possible by investigating suitably defined q-deformed Cartan–Weyl bases in the root of unity limit. Since their structure, especially for higher tensor products, is more involved we will leave this issue to future work.

Another point which deserves investigation is the finite-dimensional representation theory of the restricted quantum superalgebra at roots of unity. While for the non-graded case the representations of $U_q^{\mathrm{res}}(\hat{g})$ have been classified in [46, 47], similar results do not exist in the literature for the super case. Our discussion of the restricted quantum superalgebra $U_q^{\mathrm{res}}(\hat{g})$ is a first step in this direction.

Applying the results of our mathematical discussion, we have demonstrated that the non-deformed even subalgebra $U(\hat{g}_0)$ obtained from $U_q^{\mathrm{res}}(\hat{g})$ at roots of unity either equals or contains proper non-Abelian symmetry algebras of integrable lattice models. For instance, we established that for the special linear supergroups $\hat{g} = sl(m|n)^{(1)}$ the symmetry algebra is $sl_m \oplus sl_n \oplus u(1)$ at odd roots of unity. The associated class of integrable lattice theories includes among others the Perk–Schultz models [34] and the U-model [36, 38] which is a generalization of the Hubbard model with correlated hopping terms in the associated spin-chain Hamiltonian. Examples involving black roots only are the models based on $osp(2|2)^{(2)}$ (see e.g. [63]) whose symmetry algebra we identified to be $\hat{g}_0 = sl_2^{(1)}$.

A further step is to relate the finite-dimensional representation theory of the restricted quantum groups and the symmetry algebras to the Bethe ansatz. Since a representation independent formulation of the Bethe ansatz is not known, this can be done by choosing one of the earlier mentioned physical models, which are formulated in a specific representation of the quantum affine superalgebra.

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Appendix. Proofs

A.1. Supercommutator of the restricted algebra

The formula we like to prove for $m \geqslant n$ reads

$$\left[e_i^m, f_i^n\right] = \sum_{k=1}^{n} (-)^{|i|(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} [k]_{q_i}!\, f_i^{n-k} e_i^{m-k} \prod_{l=1}^{k} [h_i; m-n-l+1]\,.$$

Induction start:

$$\left[e_i^m, f_i\right] = [m]_{q_i} e_i^{m-1} [h_i; m-1].$$

Induction steps:

$$\begin{aligned} \left[e_i^m, f_i^{n+1}\right] &= (-)^{|i|m} f_i \left[e_i^m, f_i^n\right] + \left[e_i^m, f_i\right] f_i^n \\ &= (-)^{|i|m} f_i \left[e_i^m, f_i^n\right] + [m]_{q_i} e_i^{m-1} f_i^n [h_i; m-2n-1] \end{aligned}$$

$$\begin{aligned} \left[e_i^m, f_i^{n+1}\right] = {} & \sum_{k=1}^{n} (-)^{|i|(x_k(m,n)+m)} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} [k]_{q_i}!\, f_i^{n+1-k} e_i^{m-k} \prod_{l=0}^{k-1} [h_i; m-n-l] \\ & + \sum_{k=2}^{n+1} (-)^{|i| x_{k-1}(m-1,n)} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q_i} [k]_{q_i}! \frac{[k]_{q_i}}{[n+1]_{q_i}} f_i^{n+1-k} e_i^{m-k} \\ & \times [h_i; m-2n-1] \prod_{l=1}^{k-1} [h_i; m-n-l] \\ & + (-)^{|i|(m-1)n} [m]_{q_i} f_i^n e_i^{m-1} [h_i; m-2n-1]. \end{aligned}$$

Taking into account that $x_k(m,n) = (m-k)(n-k) \bmod 2$ one verifies the following identities:

$$\begin{aligned}
&[n+1]_{q_i}(-)^{|i|(x_k(m,n+1)-m)}[h_i; m-n-k] \\
&\qquad = (-)^{|i|x_k(m,n)}[n+1-k]_{q_i}[h_i; m-n] + (-)^{|i|(x_{k-1}(m-1,n)+m)}[k]_{q_i}\,[h_i; m-2n-1] \\
&(-)^{|i|x_1(m,n+1)}[n+1]_{q_i}[h_i; m-n-1] \\
&\qquad = (-)^{|i|x_1(m,n)}[n]_{q_i}[h_i; m-n] + (-)^{|i|(m-1)n}[h_i; m-2n-1] \\
&(-)^{|i|x_{n+1}(m,n+1)} = (-)^{|i|x_n(m-1,n)}.
\end{aligned}$$

From these equations, the desired formula for $n \to n+1$ follows, which completes the induction proof.

A.2. Proof of formula (35)

Defining $\Theta^{\pm}_{m,n}$ by means of the q-deformed adjoint action

$$\Theta^{\pm}_{m,n} := \frac{(\mathrm{ad}_{q^{\pm1}} e_i)^m e_j^n}{[m]_{q_i^{\pm1}}![n]_{q_i^{\pm1}}!}$$

we are going to prove the identity

$$\Theta^{\pm}_{m,n} = \sum_{r+s=m} (-)^{s+x_s^n} q_i^{\mp s(1-na_{ij}-m)} e_i^{(r)\pm} e_j^{(n)\pm} e_i^{(s)\pm}$$

with the degree function equal to

$$x_s^n := |i|\frac{s(s-1)}{2} + |i||j|ns.$$

Here the $\pm$ sign in the upper index of the restricted step operators refers to $q^{\pm1}$. In the following we set $y := |i|m + |i||j|n$. One then verifies by means of equation (30) that

$$\begin{aligned}
[e_i, \Theta^{\pm}_{m,n}]_{q^{\pm1}} &= e_i\Theta^{\pm}_{m,n} - (-)^y q_i^{\pm(na_{ij}+2m)}\Theta^{\pm}_{m,n}e_i \\
&= \sum_{r+s=m+1} (-)^{s+x_s} q_i^{\mp s(1-na_{ij}-m)} e_i^{(r)\pm} e_j^{(n)\pm} e_i^{(s)\pm}[r]_{q_i^{\pm1}} \\
&\quad + \sum_{r+s=m+1} (-)^{s+x_{s-1}+y} q_i^{\mp s(1-na_{ij}-m)} q_i^{\pm(m+1)} e_i^{(r)\pm} e_j^{(n)\pm} e_i^{(s)\pm}[s]_{q_i^{\pm1}} \\
&= \sum_{r+s=m+1} (-)^{s+x_s} q_i^{\mp s(-na_{ij}-m)} e_i^{(r)\pm} e_j^{(n)\pm} e_i^{(s)\pm} \Big\{ q_i^{\mp s}[r]_{q_i^{\pm1}} \\
&\quad + (-)^{x_{s-1}+x_s+y} q_i^{\pm r}[r]_{q_i^{\pm1}} \Big\} \\
&= [m+1]_{q_i^{\pm1}}\Theta^{\pm}_{m+1,n}.
\end{aligned}$$

A.3. Proof of the translation formula

We state the proof for $E_i^{(m)}$ only, the one for $F_i^{(m)}$ being completely analogous. For generic q one finds the following relations:

$$\Pi E_i \Pi^{-1} = E_i q^{H_{i;L}} + E_{i;L}(q^{-H_i} - 1)q^{H_{i;L}}$$

where use has been made of the straightforward identities

$$\begin{aligned}
&\Pi E_{i;n}\Pi^{-1} = E_{i;n-1}q^{H_{i;L}} \qquad n > 1 \\
&\Pi q^{H_{i;n}}\Pi^{-1} = q^{H_{i;n-1}}.
\end{aligned}$$

We claim that the transformation property for the mth power reads

$$\Pi E_i^m \Pi^{-1} = \sum_{n=0}^{m} (-)^{|i|\frac{n(n-1)}{2}} q_i^{n(m-1)} \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} E_i^{m-n} E_{i;L}^n q^{mH_{i;L}} \prod_{l=0}^{n-1} \left((-)^{|i|(m+l+1)} q_i^{-2l} q^{-H_i} - 1 \right).$$

Proceeding by induction, we assume that the above relation holds for m and calculate

$$\begin{aligned}\Pi E_i^{m+1} \Pi^{-1} = {} & \sum_{n=0}^{m} (-)^{|i|\frac{n(n-1)}{2}} E_i^{m-n} E_{i;L}^n E_i q_i^{n(m-1)} \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} q^{(m+1)H_{i;L}} \\ & \times \prod_{l=1}^{n} \left((-)^{|i|(m+l)} q_i^{-2l} q^{-H_i} - 1 \right) + \sum_{n=0}^{m} (-)^{|i|\frac{n(n-1)}{2}} E_i^{m-n} E_{i;L}^{n+1} q_i^{n(m-1)} \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} \\ & \times \left(q_i^{2m} q^{-H_i} - 1 \right) q^{(m+1)H_{i;L}} \prod_{l=1}^{n} \left((-)^{|i|(m+l)} q_i^{-2l} q^{-H_i} - 1 \right).\end{aligned}$$

Employing the commutation relations

$$E_{i;L}^n E_i = (-)^{|i|n} q_i^{2n} E_i E_{i;L}^n + E_{i;L}^{n+1} (1 - (-)^{|i|n} q^{2n})$$

one derives

$$\begin{aligned}\Pi E_i^{m+1} \Pi^{-1} = {} & E_i^{m+1} q^{(m+1)H_{i;L}} \\ & + (-)^{|i|\frac{(m+1)m}{2}} E_{i;L}^{m+1} q_i^{m(m-1)} q_i^{2m} q^{(m+1)H_{i;L}} \prod_{l=0}^{m} \left((-)^{|i|(m+l)} q_i^{-2l} q^{-H_i} - 1 \right) \\ & + \sum_{n=1}^{m} (-)^{|i|\frac{n(n-1)}{2}} E_i^{m+1-n} E_{i;L}^n \frac{q_i^{nm} q^{(m+1)H_{i;L}}}{[m+1]_{q_i}} \begin{bmatrix} m+1 \\ n \end{bmatrix}_{q_i} \{\cdots\} \\ & \times \prod_{l=1}^{n-1} \left((-)^{|i|(m+l)} q_i^{-2l} q^{-H_i} - 1 \right)\end{aligned}$$

where the term in the brackets reads

$$\begin{aligned}\{\cdots\} = {} & [m+1-n]_{q_i} (-)^{|i|m} q_i^{-n} q^{-H_i} + [n]_{q_i} (-)^{|i|(1-n)} q_i^{m+1-n} q^{-H_i} \\ & - [m+1-n]_{q_i} (-)^{|i|n} q_i^n - [n]_{q_i} q_i^{-m-1+n} \\ = {} & [m+1]_{q_i} ((-)^{|i|m} q^{-H_i} - 1).\end{aligned}$$

This completes the proof. Note that in the last step use has been made of the elementary relation (30) for q-deformed integers.

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